Schwarzschild black holes and propagation of electromagnetic and gravitational waves. *

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Abstract

Disturbing of a spacetime geometry may result in the appearance of an oscillating and damped radiation - the so-called quasinormal modes. Their periods of oscillations and damping coefficients carry unique information about the mass and the angular momentum, that would allow one to identify the source of the gravitational field. In this talk we present recent bounds on the diffused energy, applicable to the Schwarzschild spacetime, that give also rough estimates of the energy of excited quasinormal modes.

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1 Introduction

Quasinormal modes are scattering-type solutions of the Schrödinger equation which satisfy a quite peculiar boundary condition - that at both "ends" the waves are purely outgoing. They are being studied in the context of general relativity (the Schrödinger equation emerges there through the standard separation of the time dependence in a wave equation) in the course of the last thirty years ([1]-[3]). Much is known about their eigenvalues and their temporal evolution in the case of Schwarzschild black holes and neutron stars. An exhaustive review on that topic is [2]; see also [3]. An observer located at a fixed space position would find that quasinormal modes oscillate in time and their amplitude exponentially decays. The period of oscillations and damping coefficients carry unique information about the mass and the angular momentum.

Perturbation of a spherically symmetric spacetime geometry [4] can be described - as far as the backreaction can be neglected - by a linear wave equation ([5] - [6]). The interesting fact is that a (compact support) perturbation of a spacetime geometry may result in the appearance of an outgoing radiation that coincides in a bounded region of spacetime with a linear combination of quasinormal modes. At some intermediate time - later on the so-called tail term dominates - the perturbation is dominated by the fundamental mode, since the latter is damped weaker than the other quasinormal modes. The oscillation periods and damping coefficients do not depend on perturbations; this feature can be used in order to identify the source of the gravitational field [7].

It is of interest to know how much energy can be carried by quasinormal modes. Their energy can be estimated by the so-called diffused energy - the energy loss that is due to the backscattering. In this paper we present recent results in this direction. The order of the rest of this paper is as follows:

i) Space-time curvature and two patterns of propagation of massless fields;
ii) Vibrations of a spacetime - an example;
iii) Recent results on the energy diffusion in the Schwarzschild spacetime;
iv) Dependence of backscatter on the frequency of waves;
v) Discussion;
vi) Lessons from numerics.

2 What is backscattering?

Let an outgoing null cone $\tilde{\Gamma}_a$ originate from $(a, 0)$. Assume that a flash of radiation is initially purely outgoing [10] and that its support is contained in an annular region (a, b), $b \leq \infty$. Then, depending on whether or not the spacetime is curved, the following can be observed.

i) In the flat Minkowski spacetime a wave remains purely outgoing. No radiation can be found in the interior of the cone (Fig. 1). In this case no backscatter ([8], [9]) occurs (in the Hadamard's terminology: the type B Huyghens principle holds true).

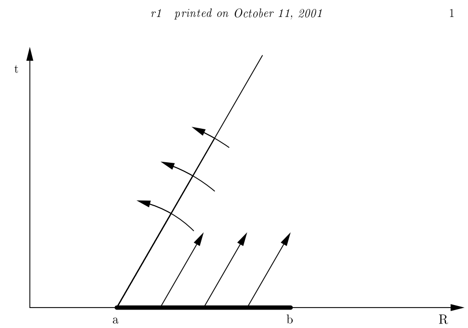

Figure 1: Solid line represents an outgoing null cone. Arrows show the direction of the radiation.

ii) A wave backscatters in a curved, e.g. Schwarzschild, spacetime. Some energy, denoted later as δE_a, diffuses inward through $\tilde{\Gamma}_a$ and is lost from the main stream (Fig 2).

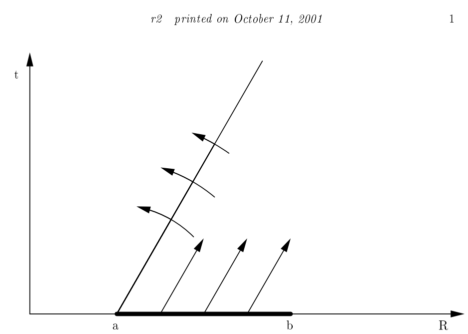

Figure 2: Solid curve represents an outgoing null cone. Arrows show the direction of the radiation - there appears an ingoing component.

One can shortly state that waves propagate in a curved spacetime like electromagnetic waves in a medium with a varying refraction index. A fraction of the radiation scatters off the curvature of the geometry and a part of the initial energy never reaches infinity.

3 Damped oscillations

Figure 3 [11] shows a picture that is typical in the case of perturbations that have compact support. An initial perturbation is depicted in Fig. 3(a). Some time after the initial pulse runs through the observer, he (or she) can observe that an oscillatory (single frequency) radiation dominates. As pointed out before, the period and damping coefficients are independent on the perturbation.

There are two interconnected problems that can made difficult the identification of the dominant mode. First, in the asymptotic zone, $t >> 2m$, the tail terms dominate [2], since they decay as some power of $1/t$. Second, the amplitude of the oscillations quickly decreases, as exemplified by Fig. 3(b).

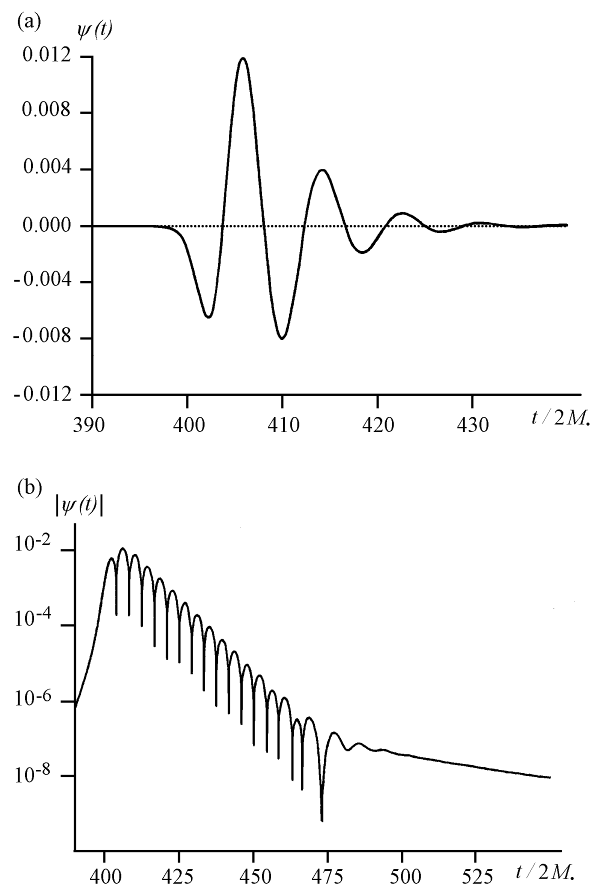

Figure 3: Time evolution of the Regge-Wheeler function for $L = 2$. The 'observer' is located at $r_* = 800 * m$

4 Quantitative estimates of the backscattered energy.

We specialize, beginning from this section, to spherically symmetric spacetimes. The backscattered (diffused) energy will be found from the energy conservation and some potential estimates. Let us point out that our intention is to derive a bound from above on the fraction of the backscattered energy; this bound should

be independent of the details of the initial pulse of the outgoing radiation. The only information that is required, is the mass m of the central object and the initial position a of the inner boundary of the outgoing radiation. (In the case of nonspherical spacetimes one presumably would need also the information about the total angular momentum.)

The idea that it is possible to bound the backscattered energy entirely in terms of initial energy and initial position was first put on trial in the example of the massless scalar field ([12] - [13]; a recent work is [14]). The Theorem formulated below summarizes results that have been obtained while implementing a programme that was formulated in the case of electromagnetic radiation in ([15], [16]) and then modified in [17] and [18]. While the analytic part concerning the electromagnetic radiation is fairly satisfactory (any further progress would most likely involve advanced numerical methods), the work on the gravitational radiation is not finished. Preliminary results on the polar modes are given in [17] while the investigation of axial modes is in progress.

Theorem.

Assume a flash of dipole electromagnetic radiation (quadrupole gravitational radiation - axial or polar) that is initially purely outgoing and that has support in an annular region (a, b), $b \leq \infty$. Let the initial energy be $E_a(0)$ and $\tilde{m} = m/a$.

Then the fraction of the diffused energy $\delta E_a / E_a(0)$ satisfies the inequality

$$\boxed{\frac{\delta E_a}{E_a(0)} \leq C(\tilde{m}) \Big(\frac{2m}{a}\Big)^2.}$$

Here

i) (the case of electromagnetic radiation) for $a \geq 10\frac{m}{3}$, $0.3 = C(0) \leq C(\tilde{m}) \leq C(10/3) \approx 1.7$

$$\begin{aligned} C(\tilde{m}) \quad = \quad & \frac{-1}{10080((-1+2\tilde{m})^4 \tilde{m}^4)} \Bigg(-2760\tilde{m}^5 + 828\tilde{m}^4 + \\ & 44\tilde{m}^3 + 2352\tilde{m}^6 + 2016\tilde{m}^4 \ln(1-2\tilde{m}) + \\ & 2688\tilde{m}^6 \ln(1-2\tilde{m}) - 4032\tilde{m}^5 \ln(1-2\tilde{m}) - \\ & 360\tilde{m}^3 \ln(1-2\tilde{m}) + 6\tilde{m} + 36\tilde{m}^2 \ln(1-2\tilde{m}) - \\ & 30\tilde{m}^2 + 3\ln(1-2\tilde{m}) - 18\tilde{m}\ln(1-2\tilde{m}) \Bigg); \end{aligned} \tag{1}$$

ii) (axial gravitational waves) [19] $1.05 = C(0) \leq C(\tilde{m}) =?$;

iii) (polar gravitational waves) [19] $55/2 = C(0) \leq C(\tilde{m}) =?$ (conjecture: $C(0) < 4$).

Sketch of the proof

4.1 Preliminaries.

A Schwarzschildean line element reads (we neglect the backreaction effect):

$$ds^2 = -(1 - \frac{2m}{R})dt^2 + \frac{1}{1 - \frac{2m}{R}}dR^2 + R^2 d\Omega^2 \ . \tag{2}$$

The radial terms of the multipole expansion satisfy the following (reduced) wave equations ([5], [6], [20]),

$$(-\partial_0^2 + \partial_{r^*}^2)\Psi_l = V(R)\Psi_l. \tag{3}$$

In the the case of lowest multipoles we have

i) $l = 1$ and $V(R) = (1 - \frac{2m}{R})\frac{2}{R^2}$ (dipole term, electromagnetism);

ii) $l = 2$ and $V(R) = (1 - \frac{2m}{R})\frac{6}{R^2}\left(1 - \frac{m}{R}\right)$ (quadrupole term, axial gravitational waves);

iii) $l = 2$ and $V(R) = (1 - \frac{2m}{R})\frac{6}{R^2}\left(1 - \frac{2m}{R} + \frac{21m^2(1+\frac{m}{R})}{4R^2(1+3m/(2R))^2}\right)$ (quadrupole term, polar gravitational waves).

In what follows we shall deal only with the electromagnetic case; the other two cases can be treated similarly. Although the calculations become more complex, basic scheme is the same.

The most general solution of the dipole electromagnetic radiation in the Minkowski spacetime is given by

$$\partial_0\Big(f(R-t) - g(R+t)\Big) + \frac{f(R-t) + g(R+t)}{R}, \tag{4}$$

where the f-related part describes the outgoing radiation while g is responsible for the ingoing wave. For any initial data, one can uniquely determine f and g and in this way specify the outgoing and ingoing initial pulses.

We invoke to the above decomposition in order to specify what is meant by in- or out- directed waves in the Schwarzschild spacetime. Define the Regge-Wheeler variable $r^*(R) \equiv R + 2m\ln(\frac{R}{2m} - 1)$. Then having initial data, one can construct functions f and g in a similar way as in the Minkowski spacetime. If from the [19] construction follows that $g = 0$, then we will say that the radiation is purely outgoing. In such a case it is useful to define

$$\tilde{\Psi}(R,t) \equiv -\partial_{r^*} f(r^* - t) + \frac{f(r^* - t)}{R(r^*)}, \tag{5}$$

and to seek a solution of the dipole wave equation, $\Psi(r^*, t)$, having the following form

$$\Psi = \tilde{\Psi} + \delta. \tag{6}$$

We would like to point out that at $t = 0$ the function δ vanishes up to the first time derivative [15], $\delta = 0$ and $\partial_0\delta = 0$. One finds that the evolution equation

reads

$$(-\partial_0^2 + \partial_{r^*}^2)\delta = (1 - \frac{2m}{R})\left[\frac{2}{R^2}\delta + \frac{6mf}{R^4}\right]. \tag{7}$$

4.2 Main steps of the proof.

The energy of the electromagnetic field that is contained in the annulus $(R(t), \infty)$ reads

$$E_{R(t)} = 2\pi \int_{R(t)}^{\infty} dr \left(\frac{(\partial_0 \Psi)^2}{1 - \frac{2m}{r}} + (1 - \frac{2m}{r})(\partial_r \Psi)^2 + \frac{2(\Psi)^2}{r^2} \right)$$

4.2.1 Bound on f in terms of the initial energy.

Lemma 1. Define $\eta_R \equiv 1 - \frac{2m}{R}$, $\tilde{m} \equiv m/a$ and $y \equiv R/a$. Let $f(a) = 0$, and $a \geq 10m/3$. Then

$$|\frac{f^2(R,0)}{R^2}| \leq \frac{E_a(0)}{8\pi} a\eta^2(R) F(\tilde{m}, y), \tag{8}$$

where

$$\begin{aligned} F(\tilde{m}, y) \equiv{}& y - 1 + \frac{16\tilde{m}^4}{3(-y + 2\tilde{m})^3} - \frac{16\tilde{m}^4}{3(-1 + 2\tilde{m})^3} - \\ & \frac{16\tilde{m}^3}{(-y + 2\tilde{m})^2} + \frac{16\tilde{m}^4}{(-1 + 2\tilde{m})^2} + \frac{24\tilde{m}^2}{(-y + 2\tilde{m})} - \\ & \frac{24\tilde{m}^2}{(-1 + 2\tilde{m})} + 8\tilde{m} \ln \frac{y - 2\tilde{m}}{1 - 2\tilde{m}}. \end{aligned} \tag{9}$$

This is essentially a Sobolev-type estimate. For the proof see [18].

4.2.2 Estimate of an "energy" of δ

This "energy" denoted as H is not conserved - there is a volume-dependent term in the integral form of this "energy" evolution law. It appears to be, however, a useful quantity. H is defined by

$$H(R,t) \equiv \int_R^{\infty} dr \Big(\frac{(\partial_0 \delta)^2}{\eta_r} + \eta_r (\partial_r \delta)^2 + \frac{2\delta^2}{r^2} \Big); \tag{10}$$

this integral is done on a fixed Cauchy hypersurface $t = const$. One can prove

Lemma 2. Let the support of initial data be (a, b), $a \geq 10m/3$, $b \leq \infty$. Then

$$H(a_t,t) \le 36m^2 \left[\int_0^t ds \left(\int_{a_s}^{\infty} dr \frac{f^2 \eta_r}{r^8} \right)^{1/2} \right]^2, \tag{11}$$

where the t-integration follows along $\tilde{\Gamma}_{a,(a_t,t)}$ while the $r-$integration is done on a fixed Cauchy slice.

The crucial point in the proof of Lemma 2 is that

$$H(a_t,t) = -\frac{\delta E_a(t)}{2\pi} - 12 \int_0^t ds \int_{a_s}^{\infty} dr \partial_0 \delta \frac{f}{r^4}; \tag{12}$$

dropping out the nonpositive δE_a terms and using the Schwarz inequality yields the lemma [18].

4.2.3 The energy conservation and bounding of the diffused energy (the energy "loss")

Define $\tilde{\Gamma}_{R_0,(R,t)}$ - a segment of an outgoing null geodesic that connects $(R_0, t = 0)$ with (R,t). Further, let us introduce

$$h_-(R,t) = \frac{1}{1-\frac{2m}{R}} (\partial_0 + \partial_{r^*}) \delta; \tag{13}$$

this corresponds to this component of the "reduced" strength field tensor that is directed inward.

The rate of the energy change along $\tilde{\Gamma}_a$ is given by

$$\begin{aligned} &(\partial_0 + \partial_{r^*}) E_a = \\ &-2\pi(1-\frac{2m}{R}) \left[(1-\frac{2m}{R}) \Big(\frac{\partial_0 \Psi}{1-\frac{2m}{R}} + \partial_R \Psi \Big)^2 + \frac{2}{R^2} \Psi^2 \right] = \\ &-2\pi(1-\frac{2m}{R}) \left[(1-\frac{2m}{R}) \left(h_- - \frac{f}{R^2} \right)^2 + \frac{2}{R^2} \Big(\tilde{\Psi} + \delta \Big)^2 \right]; \end{aligned} \tag{14}$$

but $f = \tilde{\Psi} = 0$ while their first derivatives vanish on $\tilde{\Gamma}_a$. The integration of (14) along $\tilde{\Gamma}_a$ gives the energy that diffused through the segment $\tilde{\Gamma}_{a,a(t)}$,

$$\begin{aligned} &\delta E_a(t) \equiv E_a(0) - E_a(t) = \\ &2\pi \int_a^{a(t)} dr \left[(1-\frac{2m}{r}) h_-^2 + \frac{2\delta^2}{r^2} \right]. \end{aligned} \tag{15}$$

¿From the H-conservation law and Lemma 2 one gets

$$\frac{\delta E_a}{2\pi} \le 36m^2 \left[\int_0^t ds \left(\int_{a_s}^{\infty} dr \frac{f^2 \eta_r}{r^8} \right)^{1/2} \right]^2; \tag{16}$$

here $\delta E_a \equiv \lim_{t\to\infty} \delta E_a(t)$.

The "electromagnetic" part of the main Theorem follows from the preceding bounds.

5 Dependence of the backscatter on the frequency of waves

The backscattering depends on the relative width of support. That is well known from the numerical analysis of Vishveshwara [21]), but the first - up to our knowledge - proof appeared in [16]. Below we sketch the main results. When support of the initial radiation is very narrow, $\kappa = (b-a)/a << 1$, then

$$\frac{\delta E_a}{E_a(0)} \leq C\Big(\frac{2m}{a}\Big)^2 \kappa,$$

In the limit $\kappa \to 0$ $\frac{\delta E_a}{E_a(0)} \to 0$; the backreaction is negligible. We would like to point that this argument works for all $a > 2m$, in contrast to the main Theorem.

The question arises: how do we interpret this? The *similarity theorem* [22] of Fourier transform theory states that compression of support of a function corresponds to expansion of the frequency scale. It means that

$$\lim_{\kappa\to 0} \frac{E(\omega \leq \Omega_c)}{E_a(0)} \to 0; \tag{17}$$

above $E(\omega \leq \Omega_c)$ is the electromagnetic energy of modes with frequencies smaller than a fixed frequency, $\omega \leq \Omega_c$. Thus we can conclude that the high frequency radiation is essentially unhindered by the backscattering while long waves can be backscattered.

6 Discussion

The diffused energy δE_a bounds from above the sum $\delta E_{qmt} + \delta E_f$ where δE_{qmt} is the energy carried by the quasinormal modes and the tail, and δE_f is the energy of that radiation that falls to a black hole or hits the surface of a star. Below we present several data that estimate from above the total fraction of the backscattered energy. A more detailed calculation would show that the last contribution, δE_f, dominates, so that we expect that $\delta E_{qmt}/E_a(0)$ can be a small fraction of the number given in forthcoming examples.

In the first group of examples we consider the case, when the initial pulse is close to the Schwarzschild radius $R = 2m$;

i) $a = 10m/3$ (e. g., a surface of a supercompact neutron star): $\frac{\delta E_a}{E_a(0)} < 0.5$;

ii) $a = 4m$ (e. g., a supercompact neutron star): $\frac{\delta E_a}{E_a(0)} < 0.3$;

iii) $a = 5m$ (a standard neutron star): $\frac{\delta E_a}{E_a(0)} < 0.13$.

It is instructive to notice that this effect is very weak in the case of other astronomical objects. For the Sun for instance : $\frac{\delta E_a}{E_a(0)} \approx 10^{-13}$, while for white dwarves: $\frac{\delta E_a}{E_a(0)} < 10^{-7}$. Our estimates allow one also to answer the question that was raised in the literature (see, e.g., [23]), how strongly gravitational field of a galactic interior can damp the outgoing radiation? One can infer from the preceding examples that the effect may matter only, if there is a black hole in the interior of a galaxy.

7 What can be learned from numerics?

We summarize shortly the main conclusions that can be obtained through a numerical analysis.

i) If a radiating source is close to a horizon, then the damping can be quite strong. We found, in accordance with expectations, that the backscatter is strong deep inside the photon sphere. There are known examples in which from 20 percent (electromagnetism) to 49 percent (gravitational polar modes) of the initial energy gets diffused [18]; that should go to 100 % if a very compact source is infinitely close to the event horizon.

ii) The efect depends on the width of a"renormalized" ($(b-a)/\eta_a$) support of initial energy. The backscatter is strongest, when the width is of the order of a - comparable to the areal distance from the center.

iii) The present analytic estimates are not exact. They are expected to yield, in combination with appropriate numerical methods, more satisfactory results.

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